Bargaining and Reputation

Abreu and Gul (Econometrica, 2000) summary by N. Antić and J. Tolvanen

This summary contains a detailed proof of proposition 1 and a part of proposition 4, including carefully filling in some steps missing in the original paper.

Continuous-time Bargaining

- Continuous-time bargaining, where each of two players has some probability of playing irrationally
  - the irrational type of player i plays in a specific way, identified by $\alpha_i \in (0,1)$
    - type α_i always demands α_i, accepts any offer of at least α_i and rejects remaining offers
    - Extensions to more general forms of irrationality by Abreu-Pearce (2006)
  - Assume each player i has only one irrational type
  - Let z^i denote the initial probability that i is irrational and let i's rate of discounting be r^i

Definition. *A two-player continuous-time bargaining problem is:*

$$B = \left\{ \left(\alpha_1, z^1, r^1\right), \left(\alpha_2, z^2, r^2\right) \right\},$$

where $\alpha_i \in (0,1)$, $z^i \in [0,1]$ and $r^i \in (0,1)$ for each i and $\alpha_1 + \alpha_2 > 1$. If player i concedes to player j at time t, then their respective utilities are $(1-\alpha_j)e^{-r_i t}$ and $\alpha_j e^{-r_j t}$.

- Assumption that $\alpha_1 + \alpha_2 > 1$ ensures there are no trivial solutions
- A **pure strategy** for player i is a time t_i at which he will concede
- Given t_i and t_j, the **payoff** to player i is:

$$P_i(t_i,t_j) = \begin{cases} (1-\alpha_j)\exp\left(-r^i t_i\right) & \text{if } t_i < t_j \\ \frac{1}{2}(1+\alpha_i-\alpha_j)\exp\left(-r^i t_i\right) & \text{if } t_i = t_j \\ \alpha_i \exp\left(-r^i t_j\right) & \text{if } t_i > t_j \end{cases}$$

- A **mixed strategy** for player i is a function F^i on $[0,\infty]$, where $F^i(t)$ denotes the probability of player i conceding by time t
  - **N.B.** F^i is **not** a cumulative distribution function, in partcular $\lim_{t\to\infty} F^i(t) \le 1 - z^i < 1$
- The **payoff** to player i from adopting mixed strategy F^i given that he is facing mixed strategy F^j is the usual extension of the pure strategy payoffs:

$$P_i(F^i,F^j) = \int_0^\infty \int_0^\infty P_i(t_i,t_j)dF^i(t_i)dF^j(t_j)$$

  - If a mixed strategy profile (F^1,F^2) constitutes an equilibrium, then for any $t \in supp(F^i)$ we have that:

$$P_i(t,F^j) = \sup_{s\in[0,\infty]} P_i(s,F^j)$$

Proposition (1)**.** *The **unique** sequential equilibrium of $B = \left\{\left(\alpha^1,z^1,r^1\right),\left(\alpha^2,z^2,r^2\right)\right\}$ is $\left(\widehat{F}^1,\widehat{F}^2\right)$, where:*

$$\widehat{F}^i(t) = 1 - c^i \exp\left(\frac{-r^j(1-\alpha_i)}{\alpha_j-(1-\alpha_i)}t\right),$$

where $c^i \in [0,1]$ and $(1-c^1)(1-c^2)=0$, so that at most 1 player concedes with positive probability at time 0. In fact $c^i = z^i e^{\lambda^i \tau^0}$, where:

$$\tau^0 = \min\left\{\frac{-\log z^1\left[\alpha_2-(1-\alpha_1)\right]}{r^2(1-\alpha_1)}, \frac{-\log z^2\left[\alpha_1-(1-\alpha_2)\right]}{r^1(1-\alpha_2)}\right\}.$$

- At time $\tau^0 < \infty$, the probability of irrationality of both players reaches 1 **simultaneously** and $F^i\left(\tau^0\right) = \lim_{t\to\infty} F^i(t)$.
  - If player 2 was rational and he knew that player 1 is irrational, then player 2 would concede immediately
- In the proof below, while changing the order of exposition, we kept the references to the claims made in the paper

Proof of Proposition 1. Assume that (F^1,F^2) is a sequential equilibrium of this game and let:

$$\begin{aligned} u^i(t) &= \int_0^t P_i(t,x)dF^j(x) \qquad (1) \\ &\quad + \left(F^j(t)-F^j(t^-)\right)P_i(t,t) + \left(1-F^j(t)\right)P_i(t,\infty), \end{aligned}$$

where $F^j(t^-) = \lim_{s\uparrow t} F^j(s)$, that is, $u^i(t)$ is player i's expected utility if he concedes at time t.

Claim (c) If $F^j(t^-) = F^j(t)$, then u^i is continuous at t, since:

$$\begin{aligned} \lim_{s\nearrow t}\left|u^i(t)-u^i(s)\right| &\le \lim_{s\nearrow t}\left|\int_s^t P_i(t,x)dF^j(x)\right| \\ &\quad + \lim_{s\nearrow t}\left|\left(1-F^j(t)\right)P_i(t,\infty) - \left(1-F^j(s)\right)P_i(s,\infty)\right| \\ &\le \lim_{s\nearrow t} \alpha_i\left[F(t)-F(s)\right] + 0 = 0, \end{aligned}$$

and similarly for $s \searrow t$.

Let $\tau^i = \inf\left\{t \ge 0 : F^i(t) = \lim_{s\to\infty} F^i(s)\right\}$ be the time that rational player i concedes for sure.

Claim (a) Note that $\tau^1 = \tau^2$, since a rational player will always concede once he or she knows that the other player is irrational (and will never concede). Let this time be denoted by τ^0, i.e., $\tau^0 = \tau^1 = \tau^2$.

Claim (b) For all $t > 0$ such that $F^i(t) - F^i(t^-) > 0$, we have that $F^j(t) - F^j(t^-) = 0$. This claim follows because player j would get higher expected utility by not conceding at all on the interval $(t-\varepsilon, t]$ for small $\varepsilon > 0$ and instead conceding an instant later because player i has a positive probability of conceding at time t.

Claim (d) There is no interval (s,s') such that $0 \le s < s' \le \tau^0$, $F^1(s) = F^1(s')$ and $F^2(s) = F^2(s')$.

Proof of Claim (d). Assume by way of contradiction that such an interval exists and let:

$$s^* = \sup\left\{s' \le \tau^0 : F^i(s) = F^i(s') \text{ for } i = 1,2 \text{ and } s < s'\right\}.$$

Fix $s' \in (s,s^*)$. Then for small $\varepsilon > 0$, for any $t \in (s^*-\varepsilon, s^*)$, $\exists \delta > 0$ such that :

$$u^i(s') - \delta \ge u^i(t),$$

for $i = 1,2$ (since the other player is not conceding). Note that by (b) there exists an i such that $u^i(\cdot)$ is continuous at s^*, since if $F^j(s^*) - F^j((s^*)^-) > 0$, we have that $F^i(s^*) = F^i((s^*)^-)$. Thus for this player i and for some $\eta > 0$, $u^i(t) < u^i(s')$ for any $t \in (s^*, s^*+\eta)$.

Hence F^i must be constant on $(s^*, s^*+\eta)$ since F^i is optimal. Not conceding at s' means that you shouldn't concede at any t for which $u^i(s') > u^i(t)$ (recall N.B.). But then F^j is also constant on $(s^*, s^*+\eta)$, and thus s^* could not have been the supremum as defined. ■

Claim (e) The preceding claim implies that if $0 \leq s < s' \leq \tau^0$, then $F^i(s) < F^i(s')$ for $i = 1, 2$. This follows since if F^i is always increasing, then there are no jumps in F^j.

Claim (f) Thus $F^1(t)$ and $F^2(t)$ are continuous for $t > 0$, and by (c) $u^1(t)$ and $u^2(t)$ are continuous for $t > 0$.

From (e) it follows that $A^i = \left\{t : u^i(t) = \sup_t u^i(t)\right\}$ is dense in $[0, \tau^0]$ for $i = 1, 2$. Thus u^i_t is constant for all $t > 0$, so that $A^i = (0, \tau^0]$. Importantly, $u^i(t)$ is differentiable with respect to t for $t \in \left(0, \tau^0\right)$.

Now, the fundamental theorem of calculus and the differentiability of $u^i(\cdot)$ implies that F^1 and F^2 are differentiable on $\left(0, \tau^0\right)$. Let the time derivatives be f^1 and f^2, respectively and note that by differentiating , we get the differential equation:

$$\begin{aligned}\frac{\partial}{\partial t} u^i(t) &= \frac{\partial}{\partial t} \int_0^t \alpha_i \exp\left(-r^i x\right) f^j(x) dx \\ &\quad + \frac{\partial}{\partial t} \left(1 - F^j(t)\right)(1 - \alpha_j) \exp\left(-r^i t\right),\end{aligned}$$

or:

$$\begin{aligned}0 &= \alpha_i \exp\left(-r^i t\right) f^j(t) - (1 - \alpha_j) f^j(t) \exp\left(-r^i t\right) \\ &\quad -(1 - \alpha_j) r^i \left(1 - F^j(t)\right) \exp\left(-r^i t\right).\end{aligned}$$

Re-arrange the differential equation to put it into standard form:

$$f^j(t) + \frac{(1 - \alpha_j) r^i}{[\alpha_i - (1 - \alpha_j)]} F^j(t) = \frac{(1 - \alpha_j) r^i}{[\alpha_i - (1 - \alpha_j)]}. \tag{2}$$

We will solve this linear first-order differential equation by integrating factors. The appropriate integrating factor for this equation is:

$$g(t) = \exp\left(\frac{(1 - \alpha_j) r^i}{[\alpha_i - (1 - \alpha_j)]} t\right) = \exp\left(\lambda_j t\right).$$

Multiplying both sides of equation 2 by $g(t)$ yields:

$$\begin{aligned}g(t) f^j(t) + g(t) \lambda_j F^j(t) &= g(t) \lambda_j \\ \frac{d}{dt}\left[\exp\left(\lambda_j t\right) F^j(t)\right] &= \exp\left(\lambda_j t\right) \lambda_j.\end{aligned}$$

Integrating both sides with respect to t:

$$\begin{aligned}\exp\left(\lambda_j t\right) F^j(t) &= \exp\left(\lambda_j t\right) - c^j \\ F^j(t) &= 1 - c^j \exp\left(-\lambda_j t\right),\end{aligned}$$

where c_j is a constant determined by boundary conditions. Note that by (b) if $F^i(0) > 0$ then $F^j(0) = 0$, and so at least one $c^j = 1$.

The other boundary condition comes from the fact that a rational player i would concede at time τ^0 at the latest, so that by Bayes' rule:

$$\begin{aligned}1 &= \mathbb{P}\left(\text{player } i \text{ concedes before } \tau^0 \mid \text{player } i \text{ is rational}\right) \\ &= \frac{\mathbb{P}\left(\text{player } i \text{ concedes before } \tau^0\right)}{\mathbb{P}\left(\text{player } i \text{ is rational}\right)} = \frac{F^i(\tau^0)}{1 - z^i}.\end{aligned}$$

This implies $F^i(\tau^0) = 1 - z^i$. To work out which player is the one with $c^j = 1$, solve:

$$\begin{aligned}1 - \exp\left(-\lambda_1 T^1\right) &= 1 - z^1, \text{ and} \\ 1 - \exp\left(-\lambda_2 T^2\right) &= 1 - z^2.\end{aligned}$$

The solution is:

$$T^1 = \frac{-\log z^1}{\lambda_1}, \text{ and } T^2 = \frac{-\log z^2}{\lambda_2}.$$

If $T^1 = T^2$ then both $c^1 = c^2 = 1$. Otherwise the player with $c^j = 1$ is the one with $T^j < T^i$.

The other player (whose concessions would have taken longer) makes a discrete consession at time 0, this concession is c^i. This ensures that both rational players concede for sure at the same time $\tau^0 = \min\{T^1, T^2\}$. That is, c^i is chosen so that:

$$1 - c^i \exp\left(-\lambda_i \tau^0\right) = 1 - z^i.$$

Thus $F^i = \widehat{F}^i$ for $i = 1, 2$. Since $u^j(t)$ is constant on $(0, \tau^0]$, if player i is using the above strategy F^i, then F^j is a best response (anything is). □

Discrete-time Bargaining

- The next part of the paper shows that the war of attrition is a natural limiting structure for a large family of "alternating offers"-type bargaining games
  - limit of interest is when the time between offers becomes small
- Consider the following family of discrete time bargaining games:
  - 2 players share a pie
  - If no agreement, both get zero payoff
  - If players agree in period t and player 1 is agreed to get share $x \in [0, 1]$ of the pie, players' utilities are:

    P1: $xe^{-r_1 t}$

    P2: $(1 - x)e^{-r_2 t}$,

    where r_1 and r_2 are measures of impatience
- An extensive form bargaining game is then identified with an arbitrary function $g \colon \mathbb{R}_+ \to \{0, 1, 2, 3\}$ which governs who gets to offer in which turn:
  - If $g(t) = i \in \{1, 2\}$, then player i gets to offer in turn t
  - If $g(t) = 0$, then no one gets to offer in turn t
  - If $g(t) = 3$, both get to make a simultaneous offer.
- The bargaining happens discretely
  - The set $\{t | g(t) = i \text{ or } g(t) = 3\} \cap [0, t]$ is finite for each t and $i = 1, 2$.
- Both players get to offer infinitely many times:
  - $\#\{t | g(t) = i \text{ or } g(t) = 3\} = \infty$ for $i = 1, 2$.
- The game is played as follows:
  - In turn t, if $g(t) = i \in \{1, 2\}$, player i offers a share x to player 1 and the rest to player 2
    - If the offer is accepted, the game ends and players get the utilities mentioned earlier
    - If the offer is rejected, the game continues to turn $t' = \min\{\hat{t} > t \colon g(\hat{t}) \in \{1, 2, 3\}\}$
  - In turn t, if $g(t) = 3$ both players make offers x_1 and x_2 simultaneously
    - If $x_1 \leq x_2$, then the game ends and the players get the utilities mentioned earlier, where the realized share for player 1 is $x_1 + \frac{1}{2}(x_2 - x_1)$ and for player 2 is $1 - x_2 - \frac{1}{2}(x_1 - x_2)$

- If $x_1 > x_2$, then the game continues to turn $t' = \min\{\hat{t} > t \colon g(\hat{t}) \in \{1, 2, 3\}\}$

- We can now define convergence of a sequences of discrete bargaining games

Definition. *A sequence of discrete bargaining games $(g_n)_{n\in\mathbb{N}}$ converges to continuous-time, if for all $\epsilon > 0$ there exists n_ϵ such that $n \geq n_\epsilon$ and $t \geq 0 \Rightarrow \{1, 2\} \subset g_n([t, t + \epsilon])$.*

- Informally, $(g_n)_{n\in\mathbb{N}}$ *converges* to continuous-time, if players get to offer more and more frequently in the sequence of games. Both players offer in arbitrarily small time intervals in games with high enough indices

- Strategy for player i in game n is denoted by σ_n^i and $\sigma_n = (\sigma_n^1, \sigma_n^2)$ denotes a strategy profile

- Denote the random outcome of a given profile σ_n in a discrete game g_n by $\theta_{\sigma_n} = (\bar{x}_n, \bar{t}_n)$

  - $\bar{x}_n$ is the share that player 1 receives in g_n
  - $\bar{t}_n$ is the time game g_n ends

- No agreement is identified as $\left(\frac{1}{2}, \infty\right)$

Proposition (4)**.** *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of discrete bargaining games converging to continuous-time. Let σ_n denote a sequential equilibrium of g_n and θ_{σ_n} the random outcome corresponding to σ_n. Then θ_{σ_n} converges in distribution to θ, where θ is the random outcome of the unique equilibrium from Proposition 1.*

- Proof of this is very long, these notes will only prove one lemma

Lemma. *For any $\epsilon > 0$ there exists n_ϵ such that in any sequential equilibrium of g_n for $n \geq n_\epsilon$ after any history h_t such that i is known to be rational and j is not, the payoff to i is at most $1 - \alpha_j + \epsilon$ and the payoff to j is at least $\alpha_j - \epsilon$ (evaluated at time t).*

- Informally, the lemma says that when one player is revealed to be rational, the other gets her will almost completely and almost without delay

Proof of Lemma. We want to show that if j continues to act irrationally then her payoff converges to α_j when $n \to \infty$. Assume j continues to act irrationally and $g(t) \in \{1, 2, 3\}$. Let $z_t^j = \mathbb{P}(j \text{ irrational} | j \text{ acted crazy up to time t})$. If $z_t^j \neq 0$, then Bayes' rule implies:

$$z_t^j = \frac{z_j}{\mathbb{P}(j \text{ acted crazy up to time t})} \geq z_j.$$

Since j acted irrationally up to time t, $z_t^j \neq 0$ and hence $z_t^j \geq z_j$. The game ends in finite time if j continues to act irrationally.

For t, s let:

$$P_{t|s} = (g_n \text{ doesn't end before } t | g_n \text{ hasn't ended before } s).$$

Note first that by conceding to j in period t, i can secure herself at least $1 - \alpha_j \geq (1 - \alpha_j) P_{s|t}$. Taking a strategy that may postpone the end with positive probability up to $\hat{t} + t$ can yield at most:

$$\underbrace{1 - P_{t+\widehat{t}|t}}_{\text{Get everything immediately if } j \text{ rational}} \quad + \quad \underbrace{P_{t+\widehat{t}|t} e^{-r^i \hat{t}} (1 - \alpha_j)}_{\text{wait until } t+\hat{t} \text{ and yield to irrational } j}.$$

Any strategy that may lead to such postponing can be optimal only if:

$$(1 - \alpha_j) P_{t+\widehat{t}|t} \leq 1 - P_{t+\widehat{t}|t} + P_{t+\widehat{t}|t} e^{-r^i \hat{t}} (1 - \alpha_j),$$

which holds iff $P_{t+\widehat{t}|t} \leq \frac{1}{1+(1-\alpha_j)(1-e^{-r^i\hat{t}})} \colon = \delta$. In particular if the postponing is optimal, we have that $P_{t+\widehat{t}|t} \leq \delta$. Running this argument again from time $t + \widehat{t}$ to $t + 2\widehat{t}$, we find that $P_{t+2\widehat{t}|t+\widehat{t}} \leq \delta$. By Bayes' rule:

$$\begin{aligned} \delta^2 &= \delta \cdot \delta \geq P_{t+\widehat{t}|t} \cdot P_{t+2\widehat{t}|t+\widehat{t}} \\ &= \mathbb{P}\left(g_n \text{ doesn't end before } t + 2\widehat{t} | g_n \text{ hasn't ended before } t\right) \\ &\geq z_t^j, \end{aligned}$$

since the event that j is irrational is a subset of the above event (j could for example be rational but plan on conceding at time $t + 3\widehat{t}$).

Repeating the above argument K times, we find that

$$z_t^j \leq \delta^k \to 0, \text{ as } k \to \infty,$$

but this contradicts the fact that $z_t^j > z^j$ and hence the game has to end in finite time.

Since it takes only a finite amount of time after t for g_n to end, the following function is well defined:

$$t(n) = \inf\{s \colon g_n \text{ ends before } t + s, \text{ if } j \text{ behaves irrationally}\}.$$

Note that $t(n)$ may depend on the history of play through which one arrived to period t, we supress this dependence.

We next show that for any given sequence of length t histories, $t(n) \to 0$ as $n \to \infty$. Assume the opposite. Then $(g_n)_{n\in\mathbb{N}}$ has a subsequence, WLOG assume it's $(g_n)_{n\in\mathbb{N}}$, such that there exists $(h_n^t, t(n))_{n\in\mathbb{N}}$ and an $\eta > 0$ such that game g_n ends at time $t + t(n)$ conditional on arriving there through history h_n^t and $t(n) > \eta$ for all n. Simplify notation and rescale $r^j = 1$ and $r^i = r$.

Consider the time period $[t + t(n) - \gamma, t + t(n)]$ for some $\gamma > 0$. If the game has arrived to period $t + t(n) - \gamma$ the definition of $t(n)$ implies that given j's continued irrational behavior, i must put positive probability on a strategy which leads the game to last at least γ longer.

Let x be i's expected payoff, if j agrees to an offer worse than α_j by time $\beta\gamma$ for some $\beta \in (0, 1)$. Let y be i's expected payoff, if j doesn't agree to any such offer by $\beta\gamma$. Let ξ be the probability with which i believes that j will **not** agree. Now the fact that i's strategy leads i to reject α_j implies that:

$$1 - \alpha_j \leq (1 - \xi)x + \xi y, \tag{3}$$

$$\Rightarrow \xi \leq \frac{x - 1 + \alpha_j}{x - y}, \text{ whenever } x - y > 0. \tag{4}$$

If rational j has not agreed by time $t + t(n) - \gamma + \beta\gamma$, then she knows (given i's strategy) that by waiting for $(1 - \beta)\gamma$ longer will yield her at least $\alpha_j e^{-(1-\beta)\gamma}$. This argument does not depend on $\beta < 1$, so:

$$x \leq 1 - e^{\gamma} \alpha_j. \tag{5}$$

Knowing this, player i cannot hope to get anything better than $1 - \alpha_j e^{-(1-\beta)\gamma}$ after time $t + t(n) - \gamma + \beta\gamma$, hence:

$$y \leq e^{-r\beta\gamma}(1 - e^{-(1-\beta)\gamma}\alpha_j). \tag{6}$$

We next show that for small γ and large β the right-hand side is bounded from above by $1 - \alpha_j$. To see this, notice first that:

$$e^{-r\beta\gamma}(1 - e^{-(1-\beta)\gamma}\alpha_j) < 1 - \alpha_j \iff \alpha_j < \frac{1 - e^{-r\beta\gamma}}{1 - e^{-(r\beta+(1-\beta))\gamma}}.$$

Note that both the numerator and the denominator $\to 0$ as $\gamma \to 0$. We can try l'Hôpital's rule!

The derivative of the numerator is $r\beta e^{-r\beta\gamma}$ and when $\gamma \to 0$ it approaches $r\beta$. The derivative of the denominator is $(r\beta + (1-\beta))e^{-(r\beta+(1-\beta))\gamma}$ and when $\gamma \to 0$ it approaches $(r\beta + (1-\beta))$. Thus there exists $\bar{\gamma} > 0$ s.t. for $\gamma \in (0, \bar{\gamma})$, we have that $\alpha_j < \frac{1-e^{-r\beta\gamma}}{1-e^{-(r\beta+(1-\beta))\gamma}}$ if $\alpha_j < \frac{r\beta}{r\beta+1-\beta}$. Rearanging we get that the first inequality holds if:

$$\beta > \frac{\alpha_j}{\alpha_j + r(1-\alpha_j)}.$$

Note that the right-hand side is always less than one. Thus for $\gamma < \bar{\gamma}$ and $\beta > \frac{\alpha_j}{\alpha_j+r(1-\alpha_j)}$, we have:

$$y \le e^{-r\beta\gamma}(1 - e^{-(1-\beta)\gamma}\alpha_j) < 1 - \alpha_j. \tag{7}$$

Combining (7) with (3) when the bounds for γ and β hold, we get that:

$$1-\alpha_j \le (1-\xi)x + \xi y < (1-\xi)x + \xi(1-\alpha_j)$$
$$\Rightarrow x > 1-\alpha_j > y$$

i.e., (4) holds.

Summarizing inequalities (4)-(7), for small enough γ and big enough β we have:

$$\begin{aligned} \xi &\le \frac{x-1+\alpha_j}{x-y}, & x \le 1 - e^{-\gamma}\alpha_j, \\ y &\le e^{-r\beta\gamma}(1-e^{-(1-\beta)\gamma}\alpha_j), & x \ge 1-\alpha_j > y. \end{aligned}$$

Note next that:

$$\frac{\partial}{\partial x}\left(\frac{x-1+\alpha_j}{x-y}\right) = \frac{x-y-x+1-\alpha_j}{(x-y)^2} = \frac{1-\alpha_j-y}{(x-y)^2} > 0,$$

i.e., the RHS of the first inequality is increasing in $x \Rightarrow$ substituing the RHS of the second inequality for both x yields an upper bound. Thus substituting the second and the third estimate into the first yields:

$$\xi \le \frac{\alpha_j(1-e^{-\gamma})}{1-\alpha_j e^{-\gamma} - e^{-r\beta\gamma} + \alpha_j e^{-(r\beta+(1-\beta))\gamma}}.$$

Again both the numerator and the denominator converge to zero as $\gamma \to 0$, so use l'Hôpital again. Derivative of the numerator is $\alpha_j e^{-\gamma}$ and as $\gamma \to 0$ it converges to α_j. The derivative of the denominator is $\alpha_j e^{-\gamma} + r\beta e^{-r\beta\gamma} - \alpha_j(r\beta + (1-\beta))e^{-(r\beta+(1-\beta))\gamma}$ and converges to:

$$\alpha_j + r\beta - \alpha_j(r\beta + (1-\beta)) = r\beta + \alpha_j(1-r)\beta \text{ as } \gamma \to 0.$$

Therefore $\xi \to \frac{\alpha_j}{\beta(\alpha_j+r(1-\alpha_j))}$.

Now note that, since $\beta > \frac{\alpha_j}{\alpha_j+r(1-\alpha_j)}$ we have:

$$\frac{\alpha_j}{\beta(\alpha_j + r(1-\alpha_j))} < 1.$$

Fix a β that satisfies the condition above. Then we can find $\hat{\gamma}$ s.t. $\gamma < \hat{\gamma}$ implies that $\xi < \delta$ for some $\delta < 1$. That is, the probability that player j acts irrationally for the first β fraction of the last γ units of time has to be less than δ for it to be optimal for player i to reject $1-\alpha_j$.

Now consider the remaining $(1-\beta)\gamma$ units of time. Repeat the whole argument above to get that for the β fraction of this time ξ has to be less than δ for it to be optimal to reject $1-\alpha_j$ during this time. Put together the probability that player j acts irrationally up until the last $\gamma - \beta(1-\beta)\gamma = (1-\beta)^2\gamma$ units of time must be less than δ^2 for i to reject $1-\alpha_j$.

Repeat the argument k times to get that the probability that j acts irrationally up until the last $(1-\beta)^k\gamma$ units of time must less than δ^k. Choose k such that $\delta^k < z_j$. Denote the probability that j acts irrationally up until the last $(1-\beta)^k\gamma$ units of time by ξ_k.

The event that j acts irrationally at least up to the last $(1-\beta)^k\gamma$ units of time conditional on acting irrationally up to time t is clearly a superset of the event that j acts irrationally for ever conditional on acting irrationally up to time t. Hence we get $z_t^j \le \xi_k < z_t$. But this contradicts the fact that $z_t \le z_t^j$. □

- The argument relies on the fact that someone still gets to offer during the time period $[t, t+t(n)]$ of length at least η for large n
  - If g_n doesn't converge to continuous time, this might not be true
- For each sequential equilibrium σ_n let $F_n^i\colon \mathbb{R} \to [0,1]$, where $F_n^i(t)$ is the probability that player i takes an action not consistent with being irrational in game g_n before time t conditional on the other player having played like a crazy player until time t
- To prove Proposition 4 the authors prove the following three claims
  1. Every subsequence of $(F_n^1, F_n^2)_{n\in\mathbb{N}}$ has a convergent subsequence (weak convergence).
  2. The limit points of $(F_n^1, F_n^2)_{n\in\mathbb{N}}$ do not have common points of discontinuity.
  3. If $(F_n^1, F_n^2)_{n\in\mathbb{N}}$ converges to (F^1, F^2), and F^1 and F^2 do not have common points of discontinuity, then (F^1, F^2) is an equilibrium of the continuous-time game.
- Since by Proposition 1, there is only one equilibrium in the continuous-time game, call it (F^1, F^2), every convergent subsequence of $(F_n^1, F_n^2)_{n\in\mathbb{N}}$ is converging to (F^1, F^2), and thus $(F_n^1, F_n^2)_{n\in\mathbb{N}}$ is converging to (F^1, F^2)
  - Suppose not, then there exists a subsequence $(F_{n_k}^1, F_{n_k}^2)_{k\in\mathbb{N}}$ and an $\epsilon > 0$ such that $||(F_{n_k}^1, F_{n_k}^2) - (F^1, F^2)|| > \epsilon$ for all k (in the Lévy-Prokhorov metric). However, by assumption, this subsequence has a converging subsequence, $\Rightarrow\Leftarrow$.
- Let X_i^n be a random variable that gets value t if player i reveals herself to be rational in game n at time t and zero otherwise
- By the arguments above we know that X_i^n converges in distribution to the same random variable of the continuous-time game
- Let I^n be the random variable that gets value i if player i is the first to act non-irrationally and j otherwise
  - By Continuous Mapping Theorem[1] this converges in distribution to the same thing of the continuous-time game
- By Lemma 1 the outcome from player i acting non-irrationally while j continues to act irrationally converges to $1-\alpha_j$
- Hence the outcomes of the discrete games, θ_n which are a continuous transforms of the above pointwise convergent sequence and I^ns, converge in distribution to the outcome of the continuous-time game θ.
  - Note that the division of the cake in the limit, the time of the division in the limit and the conceding player are the same as in the continuous-time game

[1] The set of discontinuity points of I^n has measure zero, this is the set $\{(x,x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. The pre-images of 1 and 2 are open.